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THE CONVERGENCE PROOF OF THE NO RESPONSE TEST FOR LOCALIZING AN INCLUSION

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Abstract - In this paper, we use the no-response test idea, introduced in [4] for the inverse obstacle problem, to identify the interface of the discontinuity of the coefficient γ of the equation $(\nabla \cdot \gamma(x)\nabla + c(x))u(x) = 0$ with piecewise regular γ and *bounded* function $c(x)$. We use infinitely many Cauchy data as measurement and give a reconstructive method to localize the interface. We give the convergence proof of the no-response test in two ways. The first one contains a pointwise estimate as used by the singular sources method. The second one is built on an energy (or an integral) estimate which is the basis of the probe method. As a conclusion of this, the no response can be seen as a unified framework for the probe and the singular sources method.

1. INTRODUCTION AND STATEMENT OF THE RESULT

1.1. Introduction

The inverse boundary value problem for identifying an inclusion inside a conductive medium from infinitely many measurements was initiated in [3]. Isakov proved uniqueness for identifying the inclusion D. Later, in [2], a method for identifying the inclusion was proposed.

For the inverse obstacle problem, in [4] the *no response test* is proposed to localize an obstacle from finitely or infinitely many measurements and in [6] we find the description of the *singular sources method* for shape reconstruction. The purpose of this paper is to use the idea of the no-response test to reconstruct the inclusion from infinitely many measurements and to clarify its relation to the probe and the singular sources method.

We show that the functional of the no-response has two different versions of lower estimates. One is of energy type. It is exactly the one of the probe method as it is given in [2]. This implies that in any case where the probe method converges then the no-response test also converges. The other version is of pointwise behavior. Its behavior is exactly the one of the singular sources method, see [6]. We will use this second version to give another convergence proof of the no-response test.

The paper is organized as follow. In the following subsection we formulate the problem and describe the no-response test for this problem. In section 2, we recall the probe method and the singular sources method in details and state the result. In the section 3, we give the idea of the proof of the result and the relations of these three methods. The complete version of this work is given in [5].

1.2 Statement of the result

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$ such that the boundary $\partial\Omega$ has the C^2 regularity. We assume that Ω contains a bounded domain D with its boundary ∂D. We suppose that ∂D has the $C^{1,1}$ regularity. We consider a function γ of the form

$$\gamma(x) \; := \; 1 + \chi_D A(x),$$

where χ_D is the characteristic function of D and $A(x)$ is a $C^1(\overline{D})$ function satisfying $A(x) > 0$ in $\overline{D}$. We denote by

$$L_\gamma := \nabla \cdot \gamma \nabla \quad \text{and} \quad M_\gamma := L_\gamma + c(x),$$

where $c(x)$ is a bounded measurable function.

Let Φ be the fundamental solution of M_1 and Φ' be the one of L_1 where M_1 and L_1 are M_γ and L_γ when $\gamma(x) = 1$, $x \in \Omega$, extended by 1 to $\mathbb{R}^n \setminus \Omega$ and $c(x)$ extended by zero to $\mathbb{R}^n \setminus \Omega$.

Further, consider $f \in H^{\frac{1}{2}}(\partial\Omega)$ and let u^f be the $H^1(\Omega)$ solution of

$$\begin{cases} M_\gamma u^f = 0 & \text{in} \quad \Omega, \\ u^f = f & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

This problem is well posed by assuming that *zero* is not an eigenvalue for the operator M_γ. By taking all the functions $f \in H^{\frac{1}{2}}(\partial\Omega)$, we define the Dirichlet to Neumann map

$$\Lambda: \ H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \ \ f \mapsto \Lambda(f) := \frac{\partial u^f}{\partial\nu}|_{\partial\Omega},$$

where ν is the exterior normal of $\partial\Omega$.

Definition (Inverse Problem.) *Let the function $c(x)$ and the Dirichlet to Neumann map Λ be known. Our task is to reconstruct the interface ∂D of discontinuity of the coefficient $\gamma(x)$.*

Now, we explain the idea of the no-response test introduced in [4] for the inverse obstacle problem and show how to adapt it to our problem.

The no-response test. By (1) and Green's formula, we write

$$\begin{aligned} u^f(x) \quad &= \quad \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y) \\ &+ \int_{\partial D} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y), \end{aligned} \tag{2}$$

for $x \in \Omega \setminus \overline{D}$. Letting $x \to \partial\Omega$ in (2) and using Green's formula, we obtain

$$\begin{aligned} u^f(x) = \frac{1}{2}u^f(x) + &\int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y) + \\ &+ \int_{\partial D} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y) \end{aligned} \tag{3}$$

for $x \in \partial\Omega$. From our Cauchy data on $\partial\Omega$, we know the function

$$J^f(x) := \frac{1}{2}u^f(x) - \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y), \quad x \in \partial\Omega.$$

By (3) we have

$$J^f(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right\} ds(y), \quad x \in \partial\Omega. \tag{4}$$

For $\varphi \in L^2(\partial\Omega)$ we define the single layer potential $v[\varphi](y)$ by

$$v[\varphi](y) = \int_{\partial\Omega} \Phi(x,y)\varphi(x)ds(x) \qquad y \in \Omega.$$

Multiplying (4) by φ, integrating over $\partial\Omega$ and exchanging the order of integration, we obtain

$$\begin{aligned} &\int_{\partial\Omega} J^f(x)\varphi(x)ds(x) \\ &= \int_{\partial\Omega} \varphi(x) \left\{ \int_{\partial D} \left(\frac{\partial u^f}{\partial\nu}(y)\Phi(x,y) - u^f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right) ds(y) \right\} ds(x) \\ &= \int_{\partial D} \left\{ \frac{\partial u^f}{\partial\nu}(y) \int_{\partial\Omega} \varphi(x)\Phi(x,y)ds(x) - u^f(y) \int_{\partial\Omega} \varphi(x)\frac{\partial\Phi(x,y)}{\partial\nu(y)}ds(x) \right\} ds(y). \end{aligned} \tag{5}$$

Hence

$$\int_{\partial\Omega} J^f(x)\varphi(x)ds(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial\nu}(y)v[\varphi] - u^f(y)\frac{\partial v[\varphi]}{\partial\nu(y)} \right\} ds(x). \tag{6}$$

Let now B be a domain inside Ω. We define the functional

$$I_{\epsilon_1,\epsilon_2}(B) \quad := \quad \sup_{(f,\varphi)\in\mathbf{M}_{\epsilon_1,\epsilon_2}(B)} \left| \int_{\partial\Omega} J^f(x)\varphi(x)ds(x) \right| \tag{7}$$

where

$$\mathbf{M}_{\epsilon_1,\epsilon_2}(B) \;:=\; \Big\{(f,\varphi)\in H^{\frac{1}{2}}(\partial\Omega)\times L^2(\partial\Omega) : \|v[\varphi]\|_{H^1(B)}\le \epsilon_1 \text{ and } \|f-v[\varphi]\|_{L^2(\partial\Omega)}\le\epsilon_2\Big\}. \tag{8}$$

Our main *indicator function* is defined by

$$I(B) \;:=\; \lim_{\epsilon_1,\epsilon_2\to 0^+} I_{\epsilon_1,\epsilon_2}(B). \tag{9}$$

Note that it is defined on a set of domains, not in the underlying 'physical' space. Now, using the data given by the Dirichlet to Neumann map we may calculate the functional (6) or the indicator function $I(B)$ defined in (7), respectively. In section 2, we give the proof of the following theorem which gives a reconstructive way how to localize ∂D.

Theorem

We have the following characterization of D from the Dirichlet to Neumann map:

$$D=\bigcap_{B\in\mathbf{B}} B,$$

where $\mathbf{B}:=\{B\subset\Omega:\ I(B)=0\}$.

2. THE PROBE AND SINGULAR SOURCES METHODS

Now we recall the probe and the singular sources methods.

The probe method. The functional of the probe method is defined by

$$\int_{\partial\Omega}(\Lambda-\Lambda_0)f(x)\cdot f(x)\,ds(x),$$

where Λ_0 is the Dirichlet-Neumann map when $\gamma=1$ in Ω.

Let now $z_p\in\Omega\setminus\overline{D}$ such that z_p tends to $z\in\Omega$ when p tends to ∞. We set $E(z_p)$ any regular domain such that $z_p\in\Omega\setminus\overline{E(z_p)}$ and $D\subset\subset E(z_p)\subset\Omega$. Using the Rungé approximation, we can find a sequence of functions, v_n^p, such that $\|\, v_n^p-\Phi(\cdot,z_p)\,\|_{H^1(E(z_p))}$ tends to zero when n tends to ∞.

We take now $f_{n,p}:=v_n^p\,|_{\partial\Omega}$ and evaluate $\int_{\partial\Omega}(\Lambda-\Lambda_0)f_n^p(x)\cdot f_n^p(x)ds(x)$, then, see [2], for every p fixed we obtain

$$\lim_{n\to\infty}\int_{\partial\Omega}(\Lambda-\Lambda_0)f_n^p(x)\cdot f_n^p(x)ds(x)=\int_D A(x)(\nabla w_p+\nabla\Phi)(x)\cdot\nabla\Phi(x)dx \tag{10}$$

where w_p is the H^1-solution of

$$\begin{cases} M_\gamma w_p=-\nabla\cdot\chi_D A(x)\nabla\Phi(\cdot,z_p) & \text{in} \quad \Omega,\\ w_p=0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

The characterization of z to be in ∂D is given by the testing

$$\lim_{p,n\to\infty}\int_{\partial\Omega}(\Lambda-\Lambda_0)f_n^p(x)\cdot f_n^p(x)ds(x)=\infty.$$

The singular sources method. For this method we take $c(x)=k^2>0$, constant. One can find a sequence of densities $g_n^p(\xi)$ such that $v_n^p:=\int_S e^{ikx\cdot\xi}g_n^p(\xi)d\xi$ tends to $\Phi(\cdot,z_p)$ in $E(z_p)$ with the H^1-norm, see [1] or [6].

We define u_n^p as the solution of

$$\begin{cases} M_\gamma u_n^p=0 \quad \text{in} \quad \Omega,\\ u_n^p=v_n^p \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then $w_n^p:=u_n^p-v_n^p$ satisfies:

$$\begin{cases} M_\gamma w_n^p=-\nabla\cdot\chi_D A(x)\nabla\Phi(\cdot,z_p) & \text{in} \quad \Omega,\\ w_n^p=u_n^p-v_n^p \quad \text{on} \quad \partial\Omega. \end{cases}$$

Tending n to ∞, we deduce that w_n^p tends to w^p in $H^1(\Omega)$ which the solution of (11). From the data $(u_n^p,\frac{\partial u_n^p}{\partial\nu})\,|_{\partial\Omega}$, we compute via the point source method the values $u_n^p(z_p)$, then we compute

$$\lim_{n\to\infty}(u_n^p(z_p)-v_n^p(z_p))=w^p(z_p). \tag{12}$$

The characterization of z to be in ∂D is given by the testing $\lim_{p\to\infty} w^p(z_p)=\infty$.

3. THE IDEA OF THE PROOF OF THE THEOREM AND A RELATION BETWEEN THE THREE METHODS

3.1 Description of the proof

We give the proof for the case $n=3$. The case $n=2$ can be treated similarly with the appropriate changes for the behavior of the related fundamental solutions. We denote by $\mathbb{N}$ the set of positive integers. We start by proving the first part of the theorem.

3.1.1 Case one

Let D be such that $D\subset B$. Let also $\varphi\in L^2(\partial\Omega)$ be such that $\|v[\varphi]\|_{H^1(B)}<\epsilon_1$ and $f\in H^{\frac{1}{2}}(\partial\Omega)$ be such that $\|f-v[\varphi]\|_{L^2(\partial\Omega)}\le\epsilon_2$. Then the function $w:=u^f-v[\varphi]\in L^2(\Omega)$ satisfies

$$\begin{cases} M_\gamma w=-\nabla\cdot A(x)\chi_D\nabla(v[\varphi]) & \text{in} \quad \Omega,\\ w=f-v[\phi] \quad \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

We decompose this function into $w:=\tilde{w}+\tilde{\tilde{w}}$ where $\tilde{w}$ satisfies (13) with homogeneous boundary condition and $\tilde{\tilde{w}}$ the solution of (13) with homogeneous equation in Ω. Hence we have $\|\tilde{w}\|_{H^1(\Omega)}\le c\epsilon_1$ and $\|\tilde{\tilde{w}}\|_{H^1(F)}\le c\epsilon_2$ for every $F\subset\subset\Omega$. Then also $\|w\|_{H^1(F)}\le c(\epsilon_1+\epsilon_2)$ for every $F\subset\subset\Omega$. Taking $F=B$, we deduce that $\|u^f\|_{H^1(B)}\le c(\epsilon_1+\epsilon_2)$. Hence $I_{\epsilon_1,\epsilon_2}(B)\le c(\epsilon_1+\epsilon_2)^2$. This means that if we have $D\subset B$, then:

$$I(B)=0.$$

3.1.2 Case two

We suppose that $\partial B\cap D\neq\emptyset$. We take a point a in $\partial D\setminus\overline{B}$ and a sequence $z_p\in\Omega\setminus(\overline{D}\cup\overline{B})$ such that z_p tends to a. We denote by $E(z_p)$ an open domain containing D and B such that $z_p\in\Omega\setminus E(z_p)$. We consider the sequence of functions $\Phi(\cdot,z_p)$. We have the following lemma.

Lemma
For every $p\in\mathbb{N}$, we can find a sequence of functions $\varphi_n^p(x)\in L^2(\partial\Omega)$ such that $\|v[\varphi_n^p]-\beta\Phi(\cdot,z_p)\|_{H^1(E(z_p))}$ tends to zero when n tends to ∞, where β is a constant independent on x.

Now for every $p\in\mathbb{N}$ fixed, we have:

$$\int_{\partial\Omega}J^f(x)\phi_n^p(x)dx=\int_{\partial D}(\frac{\partial u^f}{\partial\nu}(y)v[\varphi_n^p]-u^f(y)\frac{\partial v[\phi_n^p]}{\partial\nu(y)})ds(y).$$

For every $p\in\mathbb{N}$, let $(f_n^p)_{n,p}\subset H^{\frac{1}{2}}(\partial\Omega)$ be such that $\|f_n^p-v[\varphi_n^p]\|_{L^2(\partial\Omega)}$ tends to zero when n tends to ∞. We define $u^{f_n^p}$ the $H^1(\Omega)$-solution of the problem:

$$\begin{cases} M_\gamma u^{f_n^p}=0 \quad \text{in} \quad \Omega,\\ u^{f_n^p}=f_n^p \quad \text{on} \quad \partial\Omega. \end{cases} \tag{14}$$

Proposition 1
We have:

$$|\lim_{p,n\to\infty}\int_{\partial\Omega}J^{f_n^p}(x)\phi_n^p(x)dx|=\infty$$

By Proposition 1 and case one, we deduce the first part of the theorem. There are two ways to prove Proposition 1. In the following subsection we explain them.

3.2 Two representations for the blowup

We set $w_n^p:=u^{f_n^p}-v[\varphi_n^p]$. Hence w_n^p satisfies:

$$\begin{cases} M_\gamma w_n^p=-\nabla\cdot\chi_D A(x)\nabla v[\varphi_n^p] \quad \text{in} \quad \Omega,\\ w_n^p=f_n^p-v[\phi_n^p] \quad \text{on} \quad \partial\Omega. \end{cases} \tag{15}$$

We write $w_n^p = \tilde{w}_n^p + \tilde{\tilde{w}}_n^p$ where $\tilde{w}_n^p$ satisfies

$$\begin{cases} M_\gamma \tilde{w}_n^p = -\nabla \cdot \chi_D A(x) \nabla v[\phi_n^p] \quad \text{in} \quad \Omega, \\ \tilde{w}_n^p = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{16}$$

and $\tilde{\tilde{w}}_n^p$ is a solution of

$$\begin{cases} M_\gamma \tilde{\tilde{w}}_n^p = 0 \quad \text{in} \quad \Omega, \\ \tilde{\tilde{w}}_n^p = f_n^p - v[\phi_n^p] \quad \text{on} \quad \partial\Omega. \end{cases} \tag{17}$$

For $p \in \mathbb{N}$ fixed, we obtain that $v[\phi_n^p]$ tends to $\beta\Phi(\cdot, z_p)$ on any subset of $E(z_p)$ in the H^1 norm (see Lemma). Hence, the right hand side of (16) tends to $-\beta\nabla\cdot\chi_D A\nabla\Phi(\cdot, z_p)$ in $H^{-1}(\Omega)$. From Lax-Milgram lemma we deduce that $\tilde{w}_n^p$ is bounded in $H^1(\Omega)$ and tends weakly to some $w \in H^1(\Omega)$ which satisfies in the distribution sense $M_\gamma w = -\beta\nabla \cdot \chi_D A \nabla\Phi(\cdot, z_p)$. Similarly $\tilde{w}_n^p$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$ and, hence, converges strongly to w in $L^2(\partial\Omega)$. Thus, $w = 0$ on $\partial\Omega$.

Consider the problem (17). Since $\|f_n^p - v[\varphi_n^p]|_{L^2(\partial\Omega)}$ tends to zero as n tends to ∞, by interior estimates we deduce that $\tilde{\tilde{w}}_n^p$ tends to zero in $H^1(B)$ For all $B \subset\subset \Omega$. Finally, we deduce that w_n^p tends to $w(\cdot, z_p) \in H^1(\Omega)$ in $H^1(B)$ for every $B \subset\subset \Omega$, where w satisfies:

$$\begin{cases} M_\gamma w = -\beta\nabla \cdot \chi_D A(x) \nabla\Phi(\cdot, z_p) \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{18}$$

Since $D \subset E(z_p)$, the previous argument and Lemma imply that:

$$\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x) = \int_{\partial D} \left\{ v[\varphi_n^p] \left[\frac{\partial u^{f_n^p}}{\partial\nu} - \frac{\partial v[\varphi_n^p]}{\partial\nu} \right] + \left[v[\varphi_n^p] - u^{f_n^p} \right] \frac{\partial v[\varphi_n^p]}{\partial\nu} \right\} ds(x)$$

tends to

$$\beta \int_{\partial D} \left\{ \Phi(\cdot, z_p)\frac{\partial w}{\partial\nu} - w\frac{\partial\Phi}{\partial\nu}(\cdot, z_p) \right\} ds(x). \tag{19}$$

I. Integration in $\Omega \setminus \overline{D}$: The pointwise version of the no-response test. Using the Green's representation formula applied in $\Omega \setminus \overline{D}$, (19) becomes:

$$\beta w(z_p, z_p) - \beta \int_{\partial\Omega} \Phi(\cdot, z_p)\frac{\partial w}{\partial\nu} ds(x). \tag{20}$$

II. Integration in D: The integral version of the no-response test. We set $\Phi := \Phi(\cdot, z_p)$ and write:

$$\begin{aligned} &\int_{\partial D} \left\{ \Phi\frac{\partial w}{\partial\nu} - w\frac{\partial\Phi}{\partial\nu} \right\} ds(x) \\ = &\int_{\partial D} \left\{ \Phi\frac{\partial(w+\Phi)}{\partial\nu} - (w+\Phi)\frac{\partial\Phi}{\partial\nu} \right\} ds(x). \end{aligned} \tag{21}$$

We remark that $w + \Phi$ satisfies $\nabla \cdot \gamma\nabla(w + \Phi) = 0$ in D and recalling that $\Delta\Phi = 0$ in D we deduce that:

$$\begin{aligned} &\int_{\partial D} \left\{ \Phi\frac{\partial(w+\Phi)}{\partial\nu} - (w+\Phi)\frac{\partial\Phi}{\partial\nu} \right\} ds(x) = \\ = &-\int_{\partial D} \left\{ \Phi(1 + A(x))\frac{\partial(w+\Phi)}{\partial\nu^+} - (w+\Phi)\frac{\partial\Phi}{\partial\nu^+} \right\} ds(x) \\ = &\int_D A(x)\nabla\Phi \cdot \nabla(\Phi + w)\, dx \end{aligned} \tag{22}$$

where ν^+ is the unit normal oriented into $\Omega \setminus D$. Hence, $\int_{\partial\Omega} J^{f_n^p}(x)\phi_n^p(x)ds(x)$ tends to

$$\beta \int_D A(x)\nabla\Phi \cdot \nabla(\Phi + w)\, dx. \tag{23}$$

3.3 Outline of the Proof of Proposition 1

Now for every $\epsilon > 0$ fixed, we choose:

$$\beta := \beta(z_p, \epsilon) = \frac{\epsilon}{4} \left[max(\int_B |\Phi(\cdot, z_p)|^2 dx, \int_B |\nabla_x\Phi(\cdot, z_p)|^2 dx) \right]^{-1}.$$

With this choice, we have $\|\beta\Phi(\cdot, z_p)\|_{H^1(B)} \le \frac{\epsilon}{2}$. Since $\|v[\varphi_n^p] - \beta\Phi(\cdot, z_p)\|_{H^1(E(z_p)}$ tends to zero as n tends to ∞, for n large enough we obtain $\|v[\varphi_n^p]\|_{H^1(\overline{B})} \le \epsilon$ and $\|v[\varphi_n^p] - f_n^p\|_{L^2(\partial\Omega)} \le \epsilon$.

As a conclusion we have a sequence of functions ϕ_n^p such that for every fixed $p \in \mathbb{N}$ there is $N(p, \epsilon) \in \mathbb{N}$ such that for all $n \ge N(p, \epsilon)$ we have

$$\|v[\varphi_n^p]\|_{H^1(B)} \le \epsilon \quad \text{and} \quad \|f_n^p - v[\varphi_n^p]\|_{L^2(\partial\Omega)} \le \epsilon.$$

This sequence has the property: for p fixed, $\int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x)$ tends to

$$\beta w(z_p, z_p) - \beta \int_{\partial\Omega} \Phi(\cdot, z_p)\frac{\partial w}{\partial \nu} ds(x).$$

The function w is called the reflected solution to the system M_γ. The following proposition gives some properties of w.

Proposition 2
1) The sequence $w(z_p, z_p)$ tends to ∞ when z_p tends to a.
2) The sequence $\int_{\partial\Omega} \Phi(\cdot, z_p)\frac{\partial w}{\partial \nu} ds(x)$ is bounded with respect to p.

We remark that $\beta(z_p, \epsilon)$ is bounded with respect to z_p since $(z_p)_{p\in\mathbb{N}} \subset\subset \Omega \setminus \overline{B}$. Proposition 2 implies that

$$\lim_{p,n\to\infty} \int_{\partial\Omega} J^{f_n^p}(x)\varphi_n^p(x)ds(x) = \infty.$$

Hence $I_\epsilon(B) = \infty$. We proved the theorem.

On the other hand we can show the blowup of $\int_{\partial\Omega} J^{f_n^p}(x)\phi_n^p(x)dx$ by using (23). This is the way of the probe method, see [2].

3.4 Some comments on the relation between the three methods

The limit (23) is the one related to the probe method (10). The behavior of the pointwise estimate of the no-respose test (20) is exactly the one of the singular sources method (12). The convergence of the no-response test is a consequence of the convergence of either the probe method or the singular sources method.

Since it is known that the integral (23) diverges as z_p tends to ∂D, see [2], then $\int_{\partial\Omega} J^{f_n^p}(x)\phi_n^p(x)ds(x)$ diverges also.

To prove the convergence of these methods, the energy version is easier, see [2], since the pointwise estimates are more difficult to establish than the energy ones. Regarding the stability, the pointwise version is more suitable, see [6].

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